A geometrical approach to the dynamics of spinor condensates II: Collective modes

Ryan Barnett,[1] Daniel Podolsky,[2,3] and Gil Refael[1]

[1]*Department of Physics, California Institute of Technology, MC 114-36, Pasadena, California 91125, USA*
[2]*Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada and*
[3]*Department of Physics, Technion, Haifa, 32000, Israel*
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In this paper we study the linearized dynamics of spinor condensates using the spin-node formalism developed in Ref. [1]. We provide a general method to linearize the equations of motion based on the symmetry of the mean-field ground state using the local stereographic projection of the spin nodes. We also provide a simple construction to extract the collective modes from symmetry considerations alone akin to the analysis of vibrational excitations of polyatomic molecules. Finally, we will present a mapping between the spin-wave modes, and the wave functions of electrons in atoms, where the spherical symmetry is degraded by a crystal field.

I. INTRODUCTION

Spinor condensates have become a central theme in atomic physics since their initial creation [2–7]. Recent theoretical interests in spinor condensates have focused on topics such as dynamics near the insulating transition [8], metastable decay of currents [9], spin knots [10], and the anomalous Hall effect [11]. One particularly important aspect of spinor condensates is their free dynamics under a time-dependent Hamiltonian, about ground or metastable states. This aspect was the center of several experimental [6, 7, 12–15] and theoretical [16–22] studies. These investigations, so far, were mostly confined to the simplest case of spin-one condensates. On the other hand, the wealth and intricacy of spinor condensates increases dramatically with increasing spins. For instance, the phase diagram of spin-two and spin-three condensates consists of four, and ten possible mean-field phases, respectively [23–28]. A feature which makes these systems even more interesting, is that the ground states exhibit a high degree of symmetry in its spin state, which is isomorphic to lattice point groups [29–31]. In this paper we seek to utilize this symmetry in the study of the free dynamics of spinor condensates.

In a companion paper [1], we utilized the spin-node geometrical description of the ground state of spinor condensates and generalized it to study the hydrodynamics of these systems. Our goal was to introduce hydrodynamic equations of motion that preserved the geometrical structure of the spinor degrees of freedom, in a way that the Gross-Pitaevskii equation (GPE) cannot – the GPE requires the choice of a fixed axis of spin quantization, and it therefore hides the symmetries of the spin wave function. In Ref. [1], we expressed hydrodynamic equations of motion using the density, superfluid velocity, and the spin-nodes as our basic degrees of freedom. In addition to the Euler equations, which describe mass, momentum, and energy conservation, we obtained $2F$ Landau-Lifshitz equations for the dynamics of the spin-nodes. Furthermore, our derivation gave a natural generalization of the Mermin-Ho relation which connects the vorticity in a ferromagnetic spinor condensate with the Pontryagin index of the order parameter. We were able to

give explicit expressions for all dynamical quantities for spin-half and spin-one condensates. However, we found that the full expressions become progressively more complicated with increasing spin, and become impractical for large F.

In this paper, we study the dynamics of spinor condensates for general values of F, by looking at the low energy properties of the equations of motion obtained in Ref. 1. Our goal is to derive the spin-node description of the low lying spin-wave excitations near the mean-field ground state. In order to do this, we derive the linearized equations of motion for the spin-node locations, which allows us to extract the small oscillation spectrum from symmetry alone, in a fashion resembling the vibrational-mode calculation for polyatomic molecules [32, 33]. Using this method we are able to give simple expressions for the vibration eigenmodes and energy spectrum. In addition, we derive a correspondence between the low lying excitations of the spinor-condensates, and atomic orbitals subject to rotational symmetry due to crystal-fields, which reflect the symmetry of the spinor-condensate ground state.

The paper is organized as follows. Sec. II provides a brief summary of the main results from Ref. [1] which will be used in this paper. In Sec. III we derive the linearized equations of motion about the mean-field configuration in terms of the spin-node formalism. Finally, in Sec. IV we demonstrate how to use symmetry arguments to compute the spin-wave excitations, and give a prescription to obtain closed form expressions for both eigenmodes and eigenenergies of the low-lying spin-waves.

II. SPIN NODES AND HYDRODYNAMIC EQUATIONS

In this section, we summarize the notation and results from Ref. [1] which will be necessary for the development in this paper. We first discuss a single $F = \frac{1}{2}$ spinor and its associated vector triad, and then describe how the mean-field ground state of a spin-F condensate is described in terms of $2F$ spin nodes. Finally, we give a brief summary of the hydrodynamic equations for a

spinor condensate with general spin F.

A. Spin-half spinors and vector triads

A single $F = \frac{1}{2}$ spinor can be described in terms of a solid angle $\Omega = (\theta, \phi)$, corresponding to the direction on the unit sphere in which the spin is pointing:

$$|\Omega\rangle = \cos\left(\frac{\theta}{2}\right) e^{i\phi/2} |\uparrow\rangle + \sin\left(\frac{\theta}{2}\right) e^{-i\phi/2} |\downarrow\rangle. \quad (1)$$

This specifies the wave function uniquely, up to an overall phase. In what follows, we adhere to the phase convention above, which corresponds to a particular choice of gauge. However, the equations of motion that we will derive will be shown to be explicitly gauge invariant.

The time-reversed partner of $|\Omega\rangle$ is,

$$\left|\Omega^t\right\rangle = -\sin\left(\frac{\theta}{2}\right) e^{-i\phi/2} |\uparrow\rangle + \cos\left(\frac{\theta}{2}\right) e^{i\phi/2} |\downarrow\rangle. \quad (2)$$

which satisfies $\langle \Omega^t | \Omega \rangle = 0$. In addition, we define

$$|\Omega_x\rangle = \frac{1}{\sqrt{2}}(|\Omega\rangle + \left|\Omega^t\right\rangle) \quad (3)$$

$$|\Omega_y\rangle = \frac{1}{\sqrt{2}}(|\Omega\rangle + i\left|\Omega^t\right\rangle). \quad (4)$$

These states allow us to introduce an orthonormal triad of unit vectors $(\mathbf{n}, \mathbf{e_x}, \mathbf{e_y})$, defined by

$$\begin{aligned} \mathbf{n} &= 2\langle\Omega|\,\mathbf{F}\,|\Omega\rangle, \\ \mathbf{e}_x &= 2\langle\Omega_x|\,\mathbf{F}\,|\Omega_x\rangle, \\ \mathbf{e}_y &= 2\langle\Omega_y|\,\mathbf{F}\,|\Omega_y\rangle. \end{aligned} \quad (5)$$

Here, $\mathbf{n}$ is simply the unit vector along which the spinor points, whereas $\mathbf{e}_x$ and $\mathbf{e}_y$ are the two directions perpendicular to it. There is an ambiguity in the definition of $\mathbf{e}_x$ and $\mathbf{e}_y$ corresponding to the gauge choice in the definition of $|\Omega\rangle$. Gauge invariant quantities can be expressed in terms of $\mathbf{n}$ only, without reference to $\mathbf{e}_x$ and $\mathbf{e}_y$.

In Ref. [1] we derive a number of useful identities for the spin-half spinor and its associated triad. For instance:

$$\begin{aligned} a_\alpha &= i\,\langle\Omega|\partial_\alpha\Omega\rangle \quad (6) \\ &= \frac{1}{2}\mathbf{e}_y \cdot \partial_\alpha \mathbf{e}_x \quad (7) \end{aligned}$$

and

$$\left\langle\Omega^t\middle|\partial_\alpha\Omega\right\rangle = \frac{1}{2}\mathbf{e}_+ \cdot \partial_\alpha \mathbf{n}. \quad (8)$$

In the second expression, we have introduced the complex vectors $\mathbf{e}_\pm = \mathbf{e}_x \pm i\mathbf{e}_y$. In the first equation, we use the notation a_α to reflect the role of the quantity $i\,\langle\Omega|\partial_\alpha\Omega\rangle$ as the vector potential associated with the gauge symmetry of the spinor.

B. Spin node representation for spin-F condensate

To describe a spin-F condensate, we begin by separating the wave function into a piece corresponding to the overall density and phase, and a piece describing the local spin state. We write

$$\psi_a = \psi\chi_a \quad (9)$$

where χ_a is a normalized spin-F spinor

$$\sum_a \chi_a^* \chi_a = \langle\chi|\chi\rangle = 1 \quad (10)$$

and the superfluid density is

$$\rho = |\psi|^2. \quad (11)$$

We now turn to the spinor wave function $\chi_a = \langle a|\chi\rangle$, which is the main focus of this paper.

A general (non-normalized) spin-F spinor $|\mathbf{\Omega}\rangle$ can be represented as a totally symmetrized product of $2F$ spin-half spinors $|\Omega_1\rangle \ldots |\Omega_{2F}\rangle$ as,

$$|\mathbf{\Omega}\rangle = |\Omega_1 \ldots \Omega_{2F}\rangle = \frac{1}{\sqrt{(2F)!}} \sum_{\{\sigma\}} \left(\otimes\prod_{i=1}^{2F} |\Omega_{\sigma_i}\rangle\right) \quad (12)$$

where the sum is over all permutations σ over $2F$ indices (note that the spinor (12) is not normalized to unity, and we reserve the notation $|\chi\rangle = \frac{|\mathbf{\Omega}\rangle}{\sqrt{\langle\mathbf{\Omega}|\mathbf{\Omega}\rangle}}$ for normalized spinors). Thus, we can think of a general spin-F spinor as a collection of $2F$ indistinguishable spin-half spinors. In what follows, we call these spin-half spinors 'spin nodes'. Thus, a spin-F spinor is built out of $2F$ spin nodes, each one of which has an associated vector triad, as discussed in Sec. II A. The relationship between spin nodes and the reciprocal vectors of Ref. [29] is discussed in detail in Ref. [1].

There are a few special spinors and that will be especially useful in what follows. First, a spin-coherent state, denoted $\left|(\Omega)^{2F}\right\rangle$, is a spinor in which all spin-nodes are chosen to be the same (i.e. they all point in the same direction),

$$\left|(\Omega)^{2F}\right\rangle = |\Omega \ldots \Omega\rangle \quad (13)$$

We next define the spin state corresponding to $|\mathbf{\Omega}\rangle$ with its ith component time-reversed. We denote these by

$$|T_i\mathbf{\Omega}\rangle = \left|\Omega_1\Omega_2 \ldots \Omega_i^t \ldots \Omega_{2F}\right\rangle. \quad (14)$$

Finally, we define the projection operator $\mathcal{P}$ to be

$$\mathcal{P} = 1 - |\chi\rangle\langle\chi|. \quad (15)$$

See Ref. [1] for a more thorough discussion of the properties of the spin node representation, including a careful treatment based on the Schwinger boson approach.

C. Hydrodynamics for general spin-F condensates.

In Ref. [1] we derived the hydrodynamic equations of motion for general spin-F. The first two equations of motion, the mass continuity equation and the Euler equation are

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}) \qquad (16)$$

and

$$D_t \mathbf{v} = \mathfrak{e} + (\mathbf{v} \times \mathfrak{b}) - \nabla \left(\frac{2V_{\mathrm{int}}}{\rho} + \frac{1}{2}\Upsilon - \frac{\nabla^2 \sqrt{\rho}}{2\sqrt{\rho}} \right). \quad (17)$$

The effective electric and magnetic fields follow from the field tensor $\mathfrak{f}_{\alpha\beta}$ constructed from $a_\alpha = i \langle \chi | \partial_\alpha \chi \rangle$.

To obtain the Landau-Lifshitz equations, we contract the GPE with $\left\langle \left(\Omega_i^t \right)^{2F} \right|$. Doing this gives

$$i \langle \Omega_i^t | \partial_t \Omega_i \rangle = -\partial_\alpha \log \left(\frac{\psi}{\sqrt{\langle \mathbf{\Omega} | \mathbf{\Omega} \rangle}} \right) \langle \Omega_i^t | \partial_\alpha \Omega_i \rangle$$
$$- \frac{1}{2} \langle \Omega_i^t | \nabla^2 \Omega_i \rangle - \langle \Omega_i^t | \partial_\alpha \Omega_i \rangle \sum_{j \neq i} \frac{\langle \Omega_i^t | \partial_\alpha \Omega_j \rangle}{\langle \Omega_i^t | \Omega_j \rangle}$$
$$+ \frac{\rho}{\lambda_i^* \langle \mathbf{\Omega} | \mathbf{\Omega} \rangle} {}_1 \langle \left(\Omega_i^t \right)^{2F} |_2 \langle \mathbf{\Omega} | \mathcal{V}_{\mathrm{int}} | \mathbf{\Omega} \rangle_2 | \mathbf{\Omega} \rangle_1. \quad (18)$$

where the subscripts of the bra's and ket's in the last term denote how the inner product is to be evaluated: ket 1 (2) is contracted with bra 1 (2), and signify the state of one of two interacting particles. In this expression we have introduced the quantities λ_i,

$$\lambda_i = (2F)! \prod_{j \neq i} \langle \Omega_j | \Omega_i^t \rangle. \qquad (19)$$

See Ref. [1] for a more complete derivation of this expression.

While the first term in Eq. (18),

$$i \langle \Omega_i^t | \partial_t \Omega_i \rangle = \frac{i}{2} \mathbf{e}_{i+} \cdot \partial_t \mathbf{n}_i,$$

is the inertial term for the spin-node $\mathbf{n}_i$, the right hand side, and the last term of Eq. (18) in particular, should serve the role of torques, projected onto $\mathbf{e}_{i+}$. As we will show in the next section, the matrix element of $\mathcal{V}_{\mathrm{int}}$ is indeed related to a derivative with respect to the spin-node coordinates of a potential energy function. Specifically:

$$\frac{\rho}{\lambda_i^* \langle \mathbf{\Omega} | \mathbf{\Omega} \rangle} {}_1 \langle \left(\Omega_i^t \right)^{2F} |_2 \langle \mathbf{\Omega} | \mathcal{V}_{\mathrm{int}} | \mathbf{\Omega} \rangle_2 | \mathbf{\Omega} \rangle_1 =$$
$$A_{ij}^{-1} \rho \langle \mathbf{\Omega} | \mathbf{\Omega} \rangle \left[\mathbf{e}_{j+} \cdot \nabla_{\mathbf{n}_j} V(\{\mathbf{n}_i\}) \right], \qquad (20)$$

where $V(\{\mathbf{n}_i\}) = \langle \mathcal{V}_{\mathrm{int}} \rangle$ is the expectation value of the energy of a spin configuration with spin nodes $\{\mathbf{n}_i\}$, and A_{ij}^{-1} is a matrix which projects the torques due to spin-node j and the motion of spin-node i. The matrix A and its inverse are defined below in Eq. (29). Eq. (18) provides a natural starting point in the analysis of the linearized equations of motion which will be developed in the following section.

III. LINEARIZED EQUATIONS OF MOTION FOR ARBITRARY SPIN-F CONDENSATES

As suggested from the equations of motion of the spin-one and higher condensates in Ref. 1, the geometric representation of the equations of motion yield rather complicated results. Nevertheless, this formalism regains its appeal when linearized about particular mean-field ground states. Then the hidden point symmetries of the ground state become apparent, and can be used to describe the linearized dynamics of a condensate. Below we derive the small oscillation description of general spinor condensates.

A. Linearized equations of motion from the GPE

Parting ways from the attempt at a general description of spinor condensate dynamics, we now turn to the vicinity of a uniform mean-field ground state. For the ensuing discussion, we will denote quantities to be evaluated in the mean-field ground state with overhead bars. For instance, the density can be written by expanding about the mean-field state as

$$\rho = \bar{\rho} + \delta\rho. \qquad (21)$$

We will first concentrate on the equations describing the density excitations. To linearize the equations of motion, derived in Sec. II C, we can drop terms which involve derivatives acting on two different quantities. Doing so leads to the following two equations describing the density fluctuations:

$$\partial_t \rho = -\bar{\rho} \, \nabla \cdot \mathbf{v}, \qquad (22)$$

and

$$\partial_t \mathbf{v} = -\nabla \left(-\frac{2\bar{V}_{\mathrm{int}}}{\bar{\rho}^2} \rho + \frac{\nabla^2 \rho}{2\bar{\rho}} \right). \qquad (23)$$

Note that terms describing the spin degrees of freedom (e.g., the effective electric and magnetic fields) have completely dropped out of these equations from linearization. Computing the excitations from these equations is straightforward and gives the familiar Bogoliubov mode describing density fluctuations.

Let us now focus our attention on linearizing the Landau-Lifshitz equations for general spin written in Eq. (18). Since the process of linearization separates the equations for spin and density fluctuations, to simplify the notation in what follows, we will scale the density of the uniform state to one, $\rho_0 \to 1$. When linearized, the Landau-Lifshitz equations for general spin become

$$i \langle \bar{\Omega}_i^t | \partial_t \Omega_i \rangle = -\frac{1}{2} \langle \bar{\Omega}_i^t | \nabla^2 \Omega_i \rangle \qquad (24)$$
$$+ \frac{1}{\lambda_i^* \langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle} {}_1 \langle \left(\bar{\Omega}_i^t \right)^{2F} |_2 \langle \mathbf{\Omega} | \mathcal{V}_{\mathrm{int}} | \mathbf{\Omega} \rangle_2 | \mathbf{\Omega} \rangle_1.$$

In the above, as before, we have used overhead bars to denote quantities evaluated at their mean-field configuration.

To understand the dynamics of Eq. (24) it is useful to introduce variables to describe small deviations of the spin nodes from their mean-field values. To this end, by using the identities established in Sec. II A, we introduce the set of $2F$ complex variables $\{z_i\}$

$$z_i \equiv \langle \bar{\Omega}_i^t | \Omega_i \rangle = \langle \bar{\Omega}_i^t | \delta \Omega_i \rangle = \frac{1}{2} \bar{\mathbf{e}}_{i+} \cdot \mathbf{n}_i = \frac{1}{2} \bar{\mathbf{e}}_{i+} \cdot \delta \mathbf{n}_i, \quad (25)$$

where $\mathbf{n}_i = \bar{\mathbf{n}}_i + \delta \mathbf{n}_i$. Note that in the mean-field states we have $\bar{z}_i = 0$ for each spin node since the vectors $\bar{\mathbf{e}}_{i+}$ and $\bar{\mathbf{n}}_i$ are orthogonal. This set of variables can be seen to be the local stereographic projection of $\mathbf{n}_i$ onto the complex plane for small displacements, and will be very useful in the following analysis. Moreover, in our gauge convention, z_i is given in terms of displacements along the zenith and azimuthal directions from the spherical coordinate system:

$$z_i = \delta \mathbf{n} \cdot \hat{\theta} + i \delta \mathbf{n} \cdot \hat{\varphi}. \quad (26)$$

Using these variables, the linearized Landau-Lifshitz equations become

$$i \partial_t z_i = -\frac{1}{2} \nabla^2 z_i + \frac{{}_1\langle (\bar{\Omega}_i^t)^{2F}|\, {}_2\langle \Omega | \mathcal{V}_{\text{int}} | \Omega \rangle_2 | \Omega \rangle_1}{\lambda_i^* \langle \bar{\Omega} | \bar{\Omega} \rangle}. \quad (27)$$

The kinetic pieces in the GP equations are most naturally described in terms of the original spinor wave function, ψ_a, and are not simplified by the symmetry of the mean-field ground states. Nevertheless, Eq. (27) demonstrates that near mean-field ground states the kinetic terms still acquire a simple form. Interestingly, the kinetic parts in the spin equations of motion, (27), do not disclose the fact that the variables $\{z_i\}$ describe spin-half components of a spin-F state. This fact is reflected only in the spin interaction term. In the following section, we will see that this spin interaction can be expressed in terms of a derivative with respect to the z^* variables. In particular, the equations of motion will be shown to be

$$i \partial_t z_i = -\frac{1}{2} \nabla^2 z_i + \langle \bar{\Omega} | \bar{\Omega} \rangle \sum_j \bar{A}_{ij}^{-1} \frac{\partial}{\partial z_j^*} V_{\text{int}} \quad (28)$$

where

$$A_{ij}^{-1} \equiv \frac{\left\langle (\Omega_i^t)^{2F} \middle| (\Omega_j^t)^{2F} \right\rangle}{\lambda_i^* \lambda_j}. \quad (29)$$

Thus, the spin interaction derives from a sum over "torques,"

$$\tau_j = \frac{\partial}{\partial z_j^*} V_{\text{int}}. \quad (30)$$

B. Perturbative expansion of the spin interaction

An essential element in the behavior of spinor condensates is the spin interaction term V_{int}. It is the minimization of this term that yields the mean-field ground states, and its curvature that determines the normal excitations. These curvatures can be easily and directly extracted in terms of specific matrix elements, as we show below.

To expand the spin interaction energy about a mean-field ground state (denoted with an overhead bar) we first need to understand how to perturb a spinor about a fixed value. The following spin-half identity proves to be quite helpful:

$$|\delta \Omega \rangle = |\bar{\Omega}\rangle \langle \bar{\Omega} | \delta \Omega \rangle + |\bar{\Omega}_t \rangle z \quad (31)$$

where we used the resolution of the identity in terms of $|\bar{\Omega}\rangle$ and its time reversed partner, and the definition of $z = \langle \bar{\Omega}_t | \delta \Omega \rangle$ as in Eq. (25). Now, if we apply the variation to a general spin-F spinor $|\mathbf{\Omega}\rangle = |\bar{\mathbf{\Omega}}\rangle + \delta |\mathbf{\Omega}\rangle$, we obtain to linear order

$$\delta |\mathbf{\Omega}\rangle = |\bar{\mathbf{\Omega}}\rangle \sum_{i=1}^{2F} \langle \bar{\Omega}_i | \delta \Omega_i \rangle + \sum_{i=1}^{2F} |T_i \bar{\mathbf{\Omega}}\rangle z_i \quad (32)$$

where $|T_i \bar{\mathbf{\Omega}}\rangle$ is $|\bar{\mathbf{\Omega}}\rangle$ with its ith entry time reversed (see Sec. II B). Since $\langle \bar{\Omega}_i | \delta \Omega_i \rangle$ is imaginary, the first term, which does not directly depend on z, must drop off when considering the variations of real quantities. For instance, the first order variation of the normalization is:

$$\delta \langle \mathbf{\Omega} | \mathbf{\Omega} \rangle = \sum_{i=1}^{2F} \left(\langle \bar{\mathbf{\Omega}} | T_i \bar{\mathbf{\Omega}} \rangle z_i + \langle T_i \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle z_i^* \right). \quad (33)$$

Using Eqns. (32,33) one finds to lowest order

$$\frac{\partial}{\partial z_i} \frac{|\mathbf{\Omega}\rangle}{\langle \mathbf{\Omega} | \mathbf{\Omega} \rangle} = \frac{\bar{\mathcal{P}} |T_i \bar{\mathbf{\Omega}}\rangle}{\langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle}, \quad (34)$$

where

$$\bar{\mathcal{P}} = 1 - \frac{|\bar{\mathbf{\Omega}}\rangle \langle \bar{\mathbf{\Omega}}|}{\langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle}. \quad (35)$$

Such an expression is useful in evaluating derivatives of the spin interaction energy as in Eq. (28). In general, derivatives with respect to z_i^* will act on bras while derivatives with respect to z_i will act on kets.

We will now establish the equivalence between Eqns. (27) and (28). One can use Eq. (34) to evaluate the derivative of the interaction energy

$$\frac{\partial}{\partial z_j^*} V_{\text{int}} = \frac{{}_1\langle T_j \bar{\mathbf{\Omega}} | \bar{\mathcal{P}}_1 \, {}_2\langle \mathbf{\Omega} | \mathcal{V}_{\text{int}} | \mathbf{\Omega} \rangle_2 | \mathbf{\Omega} \rangle_1}{\langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle^2} \quad (36)$$

which is correct to linear order. The subscripts of the bra's and ket's denote how the inner product is to be evaluated: ket 1 (2) is contracted with bra 1 (2), and

signify the state of one of two interacting particles; similarly, the projection $\bar{\mathcal{P}}_1$ operates only on the degrees of freedom pertaining to particle '1'. Then using the expression for A^{-1} and the relation (derived in Appendix A)

$$\mathcal{P} = \sum_i \frac{\left|(\Omega_i^t)^{2F}\right\rangle \langle T_i \mathbf{\Omega}|}{\lambda_i} \mathcal{P} \qquad (37)$$

one immediately finds for the last term in Eq. (28)

$$\langle \bar{\mathbf{\Omega}}|\bar{\mathbf{\Omega}}\rangle \sum_j \bar{A}_{ij}^{-1} \frac{\partial}{\partial z_j^*} V_{\text{int}} = \frac{{}_1\left\langle \left(\bar{\Omega}_i^t\right)^{2F}\right|{}_2\langle \mathbf{\Omega}|\mathcal{V}_{\text{int}}|\mathbf{\Omega}\rangle_2 |\mathbf{\Omega}\rangle_1}{\lambda_i^* \langle \bar{\mathbf{\Omega}}|\bar{\mathbf{\Omega}}\rangle} \qquad (38)$$

which is the last term in Eq. (27).

1. Second order expansion of the interaction energy

Since we are interested in small oscillations about equilibrium, we would like to express the interaction energy expanded about the mean-field state to quadratic order in the z variables. This can be formally written as

$$V_{\text{int}} = \bar{V}_{\text{int}} + \frac{1}{2} \sum_{ij} \overline{\frac{\partial^2 V_{\text{int}}}{\partial z_i \partial z_j}} z_i z_j + \sum_{ij} \overline{\frac{\partial^2 V_{\text{int}}}{\partial z_i^* \partial z_j}} z_i^* z_j \quad (39)$$

$$+ \frac{1}{2} \sum_{ij} \overline{\frac{\partial^2 V_{\text{int}}}{\partial z_i^* \partial z_j^*}} z_i^* z_j^* \qquad (40)$$

where the terms involving derivatives of V_{int} are to be evaluated at the mean-field ground state. We can now use Eq. (34) to evaluate these derivatives of the interaction energy. Note that terms where two derivatives act on the same bra or ket will vanish since

$$\bar{\mathcal{P}}_1 {}_2\langle \bar{\mathbf{\Omega}}|\mathcal{V}_{\text{int}}|\bar{\mathbf{\Omega}}\rangle_2 |\bar{\mathbf{\Omega}}\rangle_1 = 0, \qquad (41)$$

which happens since $\tau_i = 0$ at the minimum of the spin interaction, so that ${}_2\langle \bar{\mathbf{\Omega}}|\mathcal{V}_{\text{int}}|\bar{\mathbf{\Omega}}\rangle_2 |\bar{\mathbf{\Omega}}\rangle_1 \propto |\bar{\mathbf{\Omega}}\rangle_1$. We then readily obtain the following quadratic form for the spin interaction energy (dropping the $\bar{V}_{\text{int}}$ term):

$$V_{\text{int}} = \sum_{ij} \left(\frac{{}_1\langle \bar{\mathbf{\Omega}}|{}_2\langle \bar{\mathbf{\Omega}}|\mathcal{V}_{\text{int}}\bar{\mathcal{P}}_2|T_i\bar{\mathbf{\Omega}}\rangle_2 \bar{\mathcal{P}}_1|T_j\bar{\mathbf{\Omega}}\rangle_1}{2\langle \bar{\mathbf{\Omega}}|\bar{\mathbf{\Omega}}\rangle^2} z_i z_j \right.$$

$$+ \frac{{}_1\langle T_i\bar{\mathbf{\Omega}}|\bar{\mathcal{P}}_1 {}_2\langle \bar{\mathbf{\Omega}}|\mathcal{V}_{\text{int}}|\bar{\mathbf{\Omega}}\rangle_2 \bar{\mathcal{P}}_1|T_j\bar{\mathbf{\Omega}}\rangle_1}{\langle \bar{\mathbf{\Omega}}|\bar{\mathbf{\Omega}}\rangle^2} z_i^* z_j$$

$$\left. + \frac{{}_1\langle T_i\bar{\mathbf{\Omega}}|\bar{\mathcal{P}}_1 {}_2\langle T_j\bar{\mathbf{\Omega}}|\bar{\mathcal{P}}_2\mathcal{V}_{\text{int}}|\bar{\mathbf{\Omega}}\rangle_2 |\bar{\mathbf{\Omega}}\rangle_1}{2\langle \bar{\mathbf{\Omega}}|\bar{\mathbf{\Omega}}\rangle^2} z_i^* z_j^* \right). \qquad (42)$$

Here $\bar{\mathcal{P}}_{1,2}$ is the projection operator which only acts on states denoted with subscripts 1 or 2 respectively. While the form above is written symmetrically, following Eq.

(41), only one projector in needed in Eq. (42), so $\mathcal{P}_2$ can be omitted.

While these results for the spin interaction seem involved, they are directly expressed in terms of easily-constructed matrix elements evaluated at the mean-field ground state. Furthermore, these matrix elements obey the point symmetry of the ground state at hand, and thus have stringent constraints. Eq. (42) therefore provides us with direct expressions for the matrix elements appearing in the linear spin-wave expansion of the spinor condensate.

C. The Lagrangian of spinor condensates near equilibrium

The equations of motion can be arrived at by expanding the spinor condensate Lagrangian to quadratic order in the z variables, and computing the corresponding Euler-Lagrange equations. As we saw before, to this order, the density excitations decouple from the spin excitations. Thus, to simplify the analysis, we will fix the density and scale it to one, and work in the incompressible regime. The Lagrangian for a spin-F condensate in the incompressible regime is

$$\mathcal{L} = a_t - \frac{1}{2}(\nabla\theta - \mathbf{a})^2 - \frac{1}{2}\Upsilon - V_{\text{int}} \qquad (43)$$

where V_{int} is the spin interaction potential. In expanding this Lagrangian to second order, we first consider the spin Berry's phase contribution

$$a_t = i\langle \chi|\partial_t\chi\rangle = \frac{i}{2}\frac{\langle \mathbf{\Omega}|\partial_t\mathbf{\Omega}\rangle - \langle \partial_t\mathbf{\Omega}|\mathbf{\Omega}\rangle}{\langle \mathbf{\Omega}|\mathbf{\Omega}\rangle}. \qquad (44)$$

Note that the kets and bras involving time derivatives are necessarily first order in variation from the mean-field state. Thus we consider the following quantity expanded to first order about the ground state

$$\delta\frac{|\mathbf{\Omega}\rangle}{\langle \mathbf{\Omega}|\mathbf{\Omega}\rangle} = \frac{|\delta\mathbf{\Omega}\rangle}{\langle \bar{\mathbf{\Omega}}|\bar{\mathbf{\Omega}}\rangle} - \frac{|\bar{\mathbf{\Omega}}\rangle}{\langle \bar{\mathbf{\Omega}}|\bar{\mathbf{\Omega}}\rangle^2}\delta(\langle \mathbf{\Omega}|\mathbf{\Omega}\rangle) \qquad (45)$$

$$= \frac{\bar{\mathcal{P}}|\delta\mathbf{\Omega}\rangle}{\langle \bar{\mathbf{\Omega}}|\bar{\mathbf{\Omega}}\rangle} - |\bar{\mathbf{\Omega}}\rangle\frac{\langle \delta\mathbf{\Omega}|\bar{\mathbf{\Omega}}\rangle}{\langle \bar{\mathbf{\Omega}}|\bar{\mathbf{\Omega}}\rangle^2}. \qquad (46)$$

Inserting this into the expression for the spin Berry's phase (44), and dropping terms that can be written as total time derivatives (which do not contribute to the dynamics) one finds

$$a_t = i\frac{\langle \mathbf{\Omega}|\bar{\mathcal{P}}|\partial_t\mathbf{\Omega}\rangle}{\langle \bar{\mathbf{\Omega}}|\bar{\mathbf{\Omega}}\rangle} \qquad (47)$$

We can then insert into Eq. (47) the expressions for the expansion of $|\mathbf{\Omega}\rangle$ to linear order in the z variables given in Eq. (32) to directly obtain

$$a_t = \frac{i}{\langle \bar{\mathbf{\Omega}}|\bar{\mathbf{\Omega}}\rangle}\sum_{ij} z_i^* \bar{A}_{ij}\partial_t z_j \qquad (48)$$

where

$$A_{ij} \equiv \langle T_i \mathbf{\Omega} | \bar{\mathcal{P}} | T_j \mathbf{\Omega} \rangle \qquad (49)$$

which is the sought-after relation. The proof that A defined here is in fact the inverse of the expression given in Eq. (29) is given in Appendix A. The hermitian matrix $\bar{A}$ gives the canonical commutation relations between the z variables. To directly compute the matrix elements of A is cumbersome because each involves a Wick expansion of $(2F)!$ terms. On the other hand the expression for A^{-1} given in Eq. (29) is readily computed since it involves evaluating overlaps between spin-coherent states. Thus, in practice, to construct the matrix A it is easiest to first construct A^{-1} and then compute its inverse.

Proceeding along very similar lines as above, one can expand Υ to second order in the z's. One finds

$$\Upsilon = \langle \partial_\alpha \chi | \bar{\mathcal{P}} | \partial_a \chi \rangle \approx \frac{1}{\langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle} \sum_{ij} \partial_\alpha z_i^* \bar{A}_{ij} \partial_\alpha z_j. \qquad (50)$$

Finally, we note that the term involving the superfluid velocity $\mathbf{v} = \nabla\theta - \mathbf{a}$ in the Lagrangian will not contribute to the linearized equations of motion. We are now in a position to vary the Lagrangian Eq. (43) as a function of the z's to find the linearized equations of motion. These read

$$i\bar{A}_{ij}\partial_t z_j = -\frac{1}{2}\bar{A}_{ij}\nabla^2 z_j + \langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle \frac{\partial V}{\partial z_i^*} \qquad (51)$$

(repeated indices are summed over). It is straightforward to see that this is the same as Eq. (28) which was obtained directly from linearizing the GPE contracted with time-reversed coherent states.

Since A is a hermitian matrix, it is diagonalized by a unitary transformation

$$A = U\Lambda U^\dagger, \qquad (52)$$

where Λ is the diagonal matrix consisting of the eigenvalues of A. It is therefore convenient to define a new set of w-coordinates as

$$w = \bar{U}^\dagger z. \qquad (53)$$

Note that in terms of these coordinates, the Berry's phase assumes a simple diagonal form

$$a_t = \frac{1}{\langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle} \sum_i \bar{\Lambda}_i w_i^* \partial_t w_i. \qquad (54)$$

Furthermore, the equations of motion have the simple form in these coordinates:

$$i\partial_t w_i = -\frac{1}{2}\nabla^2 w_i + \frac{\langle \bar{\mathbf{\Omega}} | \bar{\mathbf{\Omega}} \rangle}{\bar{\Lambda}_i} \frac{\partial V}{\partial w_i^*}. \qquad (55)$$

This has the form of a time-dependent Schrodinger equation for the w_i parameters.

IV. NORMAL EIGENMODES, SYMMETRY, AND GROUP THEORY

The most appealing application of the linearized equations of motion developed in the previous section is to obtain the normal excitation modes and energies of spinor condensates having a hidden ground state symmetry. As we show, it is nearly sufficient to diagonalize the matrix A [defined in Eq. (29)] in order to obtain the eigenmodes of the spinor-condensate. This can be done solely by using the symmetry of the hidden symmetry of the mean-field state.

Below we first demonstrate the use of the linearized equations of motion on the cyclic state without fully utilizing the symmetry in Sec. IV A, and obtain all eigenmodes and eigenfrequencies using the variables defined in Sec. III B 1. Next, in Sec. IV B, we demonstrate how from the point group of the hidden symmetry of the mean-field ground states, we can compute the normal modes alone (but not energies), using the example of the spin-three state where the spin-nodes are arranged at the vertices of a hexagon. Finally, in Sec. IV C, we show how to directly construct the vibrational and rotational eigenmodes from spherical harmonics, by connecting the problem at hand to that of degeneracy lifting of electronic atomic orbitals. This method circumvents the arduous group-theory foot work, by using the well-known properties of atomic orbitals under crystal fields that break rotational invariance.

The general motivation of the discussion below is that group theory analysis can be applied to obtain the normal modes in spinor condensates much like the analysis of the vibrational frequencies of polyatomic molecules [32, 33]. The "atoms" (or spin nodes) in our case, however, are confined to the surface of the unit sphere, and the displacement of each spin node is a two-dimensional vector (parameterized by the real and imaginary parts of the z variables). The first step in a symmetry analysis is to construct the transformation rules of the 2-d displacement vectors under point group symmetries. These transformation rules are a reducible representation of the symmetry group, and can then be broken down into its irreducible representations (irreps). The modes that transform according to the irreps are the eigenmodes of the system.

Before we begin the analysis, a note on mode multiplicity is in order. Naively, one might expect that the procedure in the previous paragraph will give $(2F) \times 2$ normal modes due to the two basis vectors per spin node. This situation would arise if the transformations we construct transform the $2F \times 2$ *real* coordinates, and are therefore 4F large *real* reducible representations of the symmetry group, resulting in 4F modes. While this is the case for real atoms, where the displacement vectors are also associated with conjugate momenta, the spin-nodes displacements do not have independent conjugate momenta. From Eq. (43) and (44) we see that the complex displacement z_i is actually canonically conjugate to

$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{z}_i} \propto i \sum_i A_{ij} z_j^*$: the two-dimensional displacements are both the coordinate and conjugate momenta, and hence there are only $2F$ eigenmodes in a spinor condensate. Qualitatively, this is a situation reminiscent of a massless particle in a magnetic field, where the x and y coordinates are canonically conjugate coordinate and momentum. Indeed, constructing real 4F dimensional representations of the symmetry would result in two duplicates of the spinor-condensate's eigenmodes. This duplicity will become evident when the eigenmodes are written in terms of the complex z_i's: half the normal modes will differ from the other half through a complex multiplicative coefficient.

A. Spin-two cyclic state

As our first example, we consider the cyclic state which is a possible mean-field ground state having the symmetry of a tetrahedron for the spin-two problem. We will expand the interaction energy to quadratic order about this mean-field ground state to compute the energies of the normal excitations. The spin-two interaction energy can be written in the simple form [25, 26]

$$V_{\text{int}} = \frac{1}{2}\alpha m^2 + \frac{1}{2}\beta |\langle \chi_t | \chi \rangle|^2 \tag{56}$$

where α and β are functions of the scattering lengths, and

$$\mathbf{m} = \langle \chi | \mathbf{F} | \chi \rangle. \tag{57}$$

For the mean-field cyclic state, this spin interaction energy conveniently vanishes $\bar{V}_{\text{int}} = 0$. In the following we will expand this energy to quadratic order.

We first construct the symmetry matrix A for the cyclic state. We take the orientation where the spin nodes are at (in cartesian coordinates)

$$\bar{\mathbf{n}}_1 = \frac{1}{\sqrt{3}}(1,1,1), \quad \bar{\mathbf{n}}_2 = \frac{1}{\sqrt{3}}(-1,-1,1), \tag{58}$$

$$\bar{\mathbf{n}}_3 = \frac{1}{\sqrt{3}}(1,-1,-1), \quad \bar{\mathbf{n}}_4 = \frac{1}{\sqrt{3}}(-1,1,-1). \tag{59}$$

With the spin-half spinors corresponding to these spin nodes the matrix $\bar{A}^{-1}$ can be directly constructed using the expression involving overlaps of time-reversed coherent states in Eq. (29). Using our gauge convention, this is found to be

$$\bar{A}^{-1} = \frac{1}{64}\begin{pmatrix} 9 & 1 & -1 & -1 \\ 1 & 9 & -1 & -1 \\ -1 & -1 & 9 & 1 \\ -1 & -1 & 1 & 9 \end{pmatrix}. \tag{60}$$

This then can be inverted to obtain

$$\bar{A} = \frac{2}{3}\begin{pmatrix} 11 & -1 & 1 & 1 \\ -1 & 11 & 1 & 1 \\ 1 & 1 & 11 & -1 \\ 1 & 1 & -1 & 11 \end{pmatrix}. \tag{61}$$

Recall that directly constructing the $\bar{A}$ matrix is cumbersome since its elements involve Wick expansions having $(2F)!$ terms. The eigenvalues of this matrix are found to be $\text{Eig}(\bar{A}) = (\bar{\Lambda}_1, \bar{\Lambda}_2, \bar{\Lambda}_3, \bar{\Lambda}_4) = (8, 8, 8, \frac{16}{3})$. This matrix can be written in a revealing form as

$$\bar{A} = 8I - \frac{8}{3}\bar{u}_4 \bar{u}_4^\dagger \tag{62}$$

where $\bar{u}_4 = \frac{1}{2}(1,1,-1,-1)^T$ is the eigenvector of $\bar{A}$ corresponding to eigenvalue $\bar{\Lambda}_4$ and I is the identity matrix. An eigenmode will necessarily diagonalize the A matrix as well as the entire equations of motion, and therefore we already gleaned one eigenmode: $\bar{u}_4$, which will turn out to be the optical mode.

The three modes orthogonal to $\bar{u}_4$ are associated with $SO(3)$ rotations. With this in mind, we construct these three eigenmodes as the vectors arising from infinitesimal rotations of $\bar{\mathbf{n}}_i$ about the cartesian axes, $\hat{x}_\alpha$. A rotation by angle $\delta\eta$ about the $\hat{\mathbf{x}}_\alpha$ axis produces the following z_i's:

$$z_i(\delta\eta) = \delta\eta\,(\hat{\mathbf{x}}_\alpha \times \bar{\mathbf{n}}_i) \cdot \bar{\mathbf{e}}_{i+} = i\delta\eta\,\bar{\mathbf{e}}_{i+} \cdot \hat{\mathbf{x}}_\alpha. \tag{63}$$

Thus the eigenvectors $\bar{u}_\alpha$ are:

$$\bar{u}_\alpha = \sqrt{\frac{3}{8}}\{\bar{\mathbf{e}}_{i+} \cdot \hat{\mathbf{x}}_\alpha\}_{i=1}^4. \tag{64}$$

It is now clear how to write the transformation into the eigen-coordinates defined generally in Eq. (53):

$$z_i = \sum_\alpha w_\alpha (\bar{u}_\alpha)_i. \tag{65}$$

Due to the high symmetry of the tetrahedron, these mode are also degenerate. In general, the set of coordinate vectors $\hat{\mathbf{x}}_\alpha$ should be taken to be the principal axes of the mean-field configuration.

Next we use this matrix to expand the interaction energy. We first consider the linear order variation of the spin moment $\mathbf{m}$. Note that since $\bar{\mathbf{m}} = 0$ in the ground state we have $\langle \bar{\Omega} | \mathbf{F} | \Omega \rangle = \langle \bar{\Omega} | \mathbf{F} \bar{\mathcal{P}} | \Omega \rangle$. Then by inserting the identity for $\bar{\mathcal{P}}$ given in Eq. (37) one finds

$$\delta\mathbf{m} = \frac{1}{2\langle \bar{\Omega} | \bar{\Omega} \rangle} \sum_{ij} (\bar{\mathbf{e}}_{i-} \bar{A}_{ij} z_j + z_i^* \bar{A}_{ij} \bar{\mathbf{e}}_{j+}). \tag{66}$$

We now write the vectors $\bar{\mathbf{e}}_{i+}$ in the basis of unit vectors along the three cartesian coordinates

$$\bar{\mathbf{e}}_{i+} = \sum_{\alpha=1}^3 (\bar{\mathbf{e}}_{i+} \cdot \hat{\mathbf{x}}_\alpha)\hat{\mathbf{x}}_\alpha \tag{67}$$

which immediately reduces them to the complex conjugate of the degenerate eigenvectors $\bar{u}_\alpha$ (with eigenvalue $\bar{\Lambda} = 8$). The fact that all of the eigenvalues are the same is due to the high symmetry of the tetrahedral state.
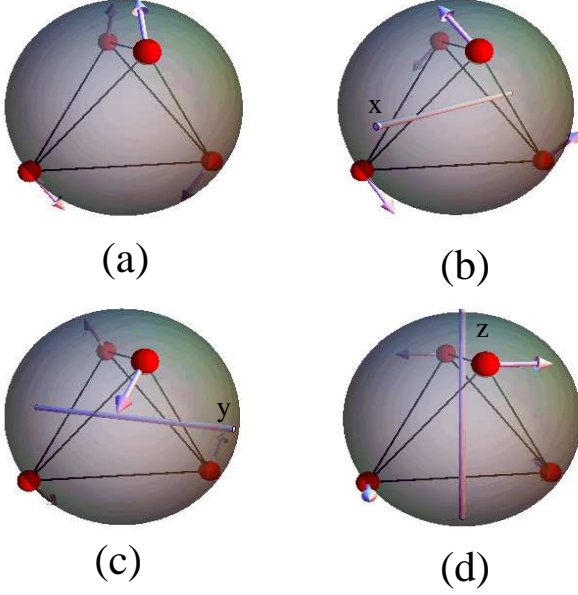
D_{6h}	E	$2C_6$	$2C_3$	C_2	$3C_2'$	$3C_2''$	i	$2S_3$	$2S_6$	σ_h	$3\sigma_d$	$3\sigma_v$
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1
A_{2g}	1	1	1	1	-1	-1	1	1	1	1	-1	-1
B_{1g}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
B_{2g}	1	-1	1	-1	-1	1	1	-1	1	-1	-1	1
E_{1g}	2	1	-1	-2	0	0	2	1	-1	-2	0	0
E_{2g}	2	-1	-1	2	0	0	2	-1	-1	2	0	0
A_{1u}	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
A_{2u}	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1
B_{1u}	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1
B_{2u}	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1
E_{1u}	2	1	-1	-2	0	0	-2	-1	1	2	0	0
E_{2u}	2	-1	-1	2	0	0	-2	1	1	-2	0	0
Γ	12	0	0	0	-4	0	0	0	0	0	0	0

TABLE I: The character table of the group D_{6h} using the notation of [32]. The last row gives the characters of the reducible representation Γ constructed from transforming the displacement vectors of the hexagon (see text).

FIG. 1: Normal modes of the cyclic state. Mode (a) is the optical mode corresponding to pure displacements in w_4. Modes (b), (c), and (d) are gapless modes corresponding rotating about the x, y, or z axes respectively. The axes of rotation for these modes is shown.

With this basis one finds for the expansion of magnetization the simple expression

$$\delta \mathbf{m} = \frac{4}{\langle \bar{\boldsymbol{\Omega}} | \bar{\boldsymbol{\Omega}} \rangle} \sum_i \sum_{\alpha=1}^{3} \sqrt{\frac{8}{3}} \hat{\mathbf{x}}_\alpha ((\bar{u}_\alpha)_i^* z_i + (\bar{u}_\alpha)_i z_i^*)$$

$$= \sqrt{6} \sum_{\alpha=1}^{3} \hat{\mathbf{x}}_\alpha (w_\alpha + w_\alpha^*) \qquad (68)$$

where we have expressed the final result in terms of the w-variables (defined in Eq. (65)). In deriving the above expression, we have explicitly used the values for the eigenvalues of the A matrix and the normalization constant $\langle \bar{\boldsymbol{\Omega}} | \bar{\boldsymbol{\Omega}} \rangle = \frac{8}{3}$. The three parameters of w_i occuring in Eq. (68) correspond to rotations about the three cartesian axes as shown in Fig. 1.

Similar analysis can be performed on the second term in the spin interaction for the cyclic state. Without showing the details, it is found that

$$\delta \langle \chi_t | \chi \rangle = 2\sqrt{2} w_4. \qquad (69)$$

With these expressions we can now write down the spin interaction energy expanded to quadratic order which reads

$$\langle \bar{\boldsymbol{\Omega}} | \bar{\boldsymbol{\Omega}} \rangle V_s = \alpha \sum_{i=1}^{3} \bar{\Lambda}_i (w_i + w_i^*)^2 + 2\beta \bar{\Lambda}_4 |w_4|^2. \qquad (70)$$

With this expansion of the interaction, Eq. (55) can be directly used to compute the energy of the normal excitations. Four Bogoliubov modes (note we are neglecting the

density mode) are readily obtained. One finds three gapless spin waves of dispersion $E_k^s = \sqrt{\varepsilon_k(\varepsilon_k + 4\alpha)}$ in addition to an optical mode having dispersion $E_k^{op} = \varepsilon_k + 2\beta$ (where ε_k is the free particle dispersion).

Quite generally, the eigenvectors of the matrix $\bar{A}$ yield the displacements of the z variables corresponding to each of the eigenmodes (see, e.g., Eq. (63). In case of degeneracy, it is the interaction terms, discussed in Sec. III B 1, that determine the correct diagonalization of the degenerate subspace in the matrix A. In the case of the cyclic state, the first three modes have displacements that correspond to rotations about three orthogonal axes. The final mode $z \propto \bar{u}_4$ corresponds to the optical excitation discussed above, and its displacements are depicted in Fig. 1. This procedure simplifies the standard Bogoliubov method [34] considerably; we extract the eigenmodes solely from the A matrix, which, as we show next, can be obtained from symmetry considerations.

B. Spin-three hexagonal state

Let us now describe how to obtain the normal modes of a spinor condensate by using symmetry arguments alone in a more complicated setting. Once having the eigenmodes, however, we must note that to obtain the energetics and dispersions of these modes, analysis of the microscopic Hamiltonian is still required. Our analysis uses group theoretical arguments similar to those used to determine the vibrational modes of polyatomic molecules [32, 33]. We illustrate the method through the nontrivial example of the spin-three state having the symmetry of the hexagon, which is a candidate for the ground state of ^{52}Cr condensates [27, 28].

The hexagon belongs to the point symmetry group D_{6h} whose character table is given in Table I. In this table we
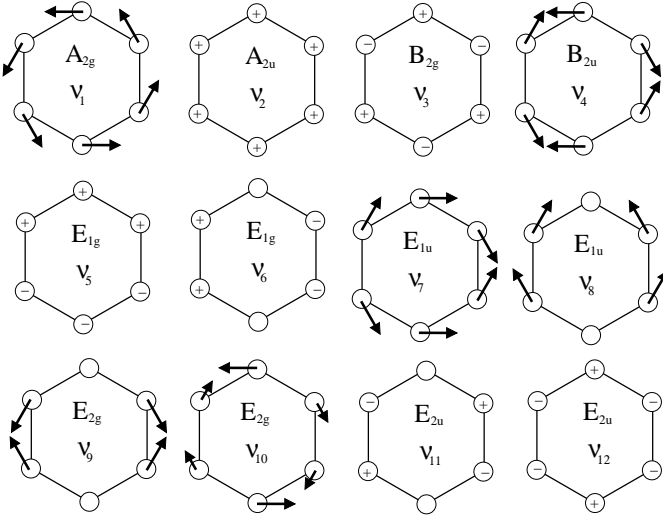
FIG. 2: Normal modes for the hexagonal configuration of the spin-three condensate. Vectors moving into and out of the plane are denoted with "-" and "+" respectively. By multiplying the set of parameters $\{z_i\}$ corresponding to these displacements by a factor of i, one can identify $\nu_1 = \nu_2$, $\nu_3 = \nu_4$, $\nu_5 = \nu_7$, $\nu_6 = \nu_8$, $\nu_9 = \nu_{11}$, and $\nu_{10} = \nu_{12}$. The modes $\nu_1 = \nu_2$, $\nu_5 = \nu_7$, and $\nu_6 = \nu_8$, correspond to Goldstone excitations due to the broken spin symmetry, while all other modes are optical and gapped. The mode $\nu_{10} = \nu_{12}$ has a set of displacement vectors with lengths differing by a factor of two.

use the standardized notation for the symmetry operators and irreducible representations [32]. To every spin node, we attach two displacement vectors parameterized by the real and imaginary parts of the z_i's introduced previously. Such displacement vectors are always parallel to the surface of the sphere. We can construct matrices M_i which describe how this set of $2 \cdot 2F = 4F$ vectors transform under each of the symmetry operations. It is easy to then see that this set of matrices $\Gamma \equiv \{M_i\}$ form a (reducible) representation of the symmetry group. While these large $4F \times 4F$ matrices are cumbersome to write down, their characters (traces) can be obtained by inspection. For instance, only spin nodes which are mapped to themselves by a particular symmetry operation will contribute to the character of the matrix describing this symmetry operation. The last row of Tab. I gives the characters of each of the matrices M_i forming Γ.

One can then invert the character matrix given in Tab. I to see how Γ can be decomposed into combinations of irreducible representations. The result is

$$\Gamma = A_{2g} + B_{2g} + E_{1g} + E_{2g} + A_{2u} + B_{2u} + E_{1u} + E_{2u}. \quad (71)$$

In the typical notation [32] A's and B's denote one-dimensional irreducible representations while E's denote two-dimensional irreducible representations. The normal modes form the basis of each of these irreducible representations [32]. For two-dimensional irreducible representations, there is some ambiguity in picking the two

basis functions. For simplicity, we picked the particular displacements which are all in-plane or all out-of-plane to form such bases. The $4F$ modes corresponding to each of these representations is given in Fig. 2. As usual, the modes corresponding to two-dimensional representations are degenerate.

Once we know the irreducible representations involved, we follow standard group theory, and construct projection operators for the modes in these irreducible representations. A general displacement of the spin nodes Q can be decomposed into a superposition of modes forming bases for each irreducible representation as

$$Q = \sum_i \mathcal{P}(\Gamma_i) Q \quad (72)$$

where the operator $\mathcal{P}(\Gamma_i)$ projects into the irreducible representation Γ_i. Such projection operators can be written explicitly as

$$\mathcal{P}(\Gamma_i) = \frac{\ell_i}{h} \sum_g \chi^{(\Gamma_i)}(g) \ D(g). \quad (73)$$

Here, $\chi^{(\Gamma_i)}(g)$ is the character for the irreducible representation Γ_i corresponding to group element g, ℓ_i is the dimension of the ith irreducible representation, and h are the number of elements in the symmetry group; $D(g)$ is the representation of group element g in the spin-nodes displacement basis. For the hexagonal state of the spin-three condensate, this projection confirms the eigenmodes depicted in Fig. 2.

As mentioned above, unlike molecular normal modes where the atoms oscillate linearly about the equilibrium positions, the spin nodes will rotate along ellipses about the equilibrium configuration. This allows us to cut the number of modes given Fig. 2 in half. Specifically, by multiplying the displacements $\{z_i\}$ by the phase factor of i, we identify $\nu_1 = \nu_2$, $\nu_3 = \nu_4$, $\nu_5 = \nu_7$, $\nu_6 = \nu_8$, $\nu_9 = \nu_{11}$, and $\nu_{10} = \nu_{12}$. Because of rotational invariance, the aspect ratio of the ellipses for the three spin rotational Goldstone modes $\nu_1 = \nu_2$, $\nu_5 = \nu_7$, and $\nu_6 = \nu_8$ will be zero. Finally, we identify the three remaining modes $\nu_3 = \nu_4$, $\nu_9 = \nu_{11}$, and $\nu_{10} = \nu_{12}$ with gapped optical modes of the hexagonal spin-three condensate.

Thus, for this spin-three problem, by symmetry arguments alone we have identified the $2F = 6$ spin modes (three of which are Goldstone modes). These modes along with density mode give the complete spectrum of normal modes for the spin-three hexagonal condensate.

C. Connection to atomic orbital theory and spherical harmonics

The treatment above makes the construction of low-energy eigenmodes of spinor condensates geometrically intuitive, and illustrates how to directly use the machinery of group theory. In addition, however, it is possible to make use of the close relationship of the symmetry

group and the underlying full $SO(3)$ rotational symmetry (such a connection was explored in the context of equilibrium spinor-condensates in Ref. [30]). Once this connection is made, we will be able to simply map the already well-developed theory of crystal-field splittings of atomic orbitals to the problem of eigenmodes of spinor condensates.

The connection between the z-representation of small oscillations as in Sec. IV A and spherical harmonics can be deduced from the transformation rules of the vector z_i under the relevant point-group. On the one hand, a symmetry operator in the z-representation, $D_{ij}(g)$, will permute the entries z_i, as the symmetry operation g would the spin-nodes. On the other hand, each z_i is a two-dimensional vector written in terms of a complex number with respect to a particular basis pair, $\bar{\mathbf{e}}_x, \bar{\mathbf{e}}_y$, which are functions of the location of the spin-node $\mathbf{n}$ on the unit sphere. Therefore, the operator $D_{ij}(g)$ also contains phase factors, $e^{i\lambda_{ij}(g)}$, which serve to rotate the basis vectors. So, in general, the structure of symmetry operators in the z-basis is

$$D_{ij}(g) = A_{ij}^{(2F)}(g)e^{i\lambda_{ij}(g)}, \qquad (74)$$

where $A_{ij}^{(2F)}(g)$ is an element of the $2F$ permutation group corresponding to a rotational symmetry of the spin nodes.

By exploiting the above transformation structure, we can systematically construct bases of the symmetry group over $\mathbf{C}^{2F}$ from the bases of rotational symmetry, namely, spherical harmonics, $Y_{lm}(\theta, \phi)$. Let us mark the polar coordinates of the spin node $\mathbf{n}_i$ as θ_i, ϕ_i; from this set of coordinates, we can produce a $2F$-dimensional complex vector:

$$\{Y_{lm}(\theta_i, \phi_i)\}_{i=1}^{2F}.$$

It is easy to see that if we apply a rotational symmetry operator g of the spinor condensate on this vector, we have:

$$\sum_{m'} R_{mm'}^{(l)}(g)Y_{lm'}(\theta_i, \phi_i) = \sum_j A_{ij}^{(2F)}(g)Y_{lm}(\theta_j, \phi_j), \qquad (75)$$

where $A_{ij}^{(2F)}(g)$ is the permutation operator from Eq. (74). The right-hand side of this equation indicates the rearrangement of the spin-nodes due to the symmetry operator. On the other hand, the left-hand side comes from our knowledge of the transformation rules for spherical harmonics, under rotations: namely, l, the total angular momentum is invariant, and the different azimuthal angular momentum components mix under the transformation.

To connect the spherical harmonics with the z-representation, we need to construct a vector that will also transform with the phase $e^{i\lambda_{ij}(g)}$. This requires that in addition to evaluating the spherical harmonics at the points (θ_i, ϕ_i), we need to account for the phase

factor when constructing the derived bases in the z-representation. This can be achieved by the following notion: instead of looking at the value of $Y_{lm}(\theta, \phi)$, let us look at its derivative, which in our gauge convention can be written as

$$\frac{\partial Y_{lm}(\theta, \phi)}{\partial z^*} = \left(\frac{\partial}{\partial \theta} + i\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right) Y_{lm}(\theta, \phi). \qquad (76)$$

The denominator of the partial derivative ∂z_i^* can be thought of as a small deviation, δz_i^*, from the mean-field spin node; it obeys the complex conjugate of the transformation rule in Eq. (74), so its inverse transforms in the correct way, using the phase $e^{i\lambda_{ij}(g)}$. Therefore, we finally have the connection between the symmetry of the spinor condensate and the representations of $SO(3)$:

$$\sum_{m'} R_{mm'}^{(l)}(g)\frac{\partial Y_{lm'}(\theta_i, \phi_i)}{\partial z_i^*} = \sum_j D_{ij}(g)\frac{\partial Y_{lm}(\theta_j, \phi_j)}{\partial z_j^*}. \qquad (77)$$

What we achieved by making this connection is of constructing for each $l > 1$ ($l = 0$ gives identically zero) partially reduced (albeit still reducible) representations of the symmetry group at hand in terms of the z-parametrization of small deviations from equilibrium. Let us denote the vectors we construct from Y_{lm} as:

$$\mathbf{u}_{l,m} = \left\{\frac{\partial Y_{lm}(\theta_i, \phi_i)}{\partial z_i^*}\right\}_{i=1}^{2F}. \qquad (78)$$

These vectors are the simplest building blocks for the vibrational and rotational eigenvectors.

As an example, consider the $l = 1$ states (p-states) obtained for the cyclic state of spin-two condensates. With the orientation for the cyclic state given in Sec. IV A, we obtain for $l = 1, m = 0$:

$$\mathbf{u}_{1,0} \propto \{\bar{\mathbf{e}}_{i+} \cdot \hat{x}_3\}_{i=1}^4 \propto (1, 1, 1, 1) \qquad (79)$$

which is the displacement vector for $x_3 = z$-axis rotation, as in Eq. (63). For $m = \pm 1$, as in atomic-orbital physics, it is useful to construct the p_x and p_y combinations, which are $p_{x,y} \propto Y_{1,1} \mp Y_{1,-1}$. For p_x we obtain:

$$\mathbf{u}_{1,1} - \mathbf{u}_{1,-1} \propto \{\bar{\mathbf{e}}_{i+} \cdot \hat{x}_1\}_{i=1}^4 \propto \{1, -1, -1, 1\}, \qquad (80)$$

which is the rotation about the $x_1 = x$-axis (up to an overall complex coefficient). In the same fashion we find that the p_y combination is

$$\mathbf{z}_{1,1} + \mathbf{z}_{1,-1} \propto \{\bar{\mathbf{e}}_{i+} \cdot \hat{x}_2\}_{i=1}^4 \propto \{1, -1, 1, -1\} \qquad (81)$$

which is corresponds to rotation about the $x_2 = y$-axis. To obtain the last mode, which is the optical vibration mode shown in Fig. 1, all we need is to find the vector of $\mathbf{u}$ which is orthogonal to the above three.

The above analogy with atomic p-orbitals is not accidental. Since we mapped vibrational modes to spherical harmonics, we also mapped the z-representation of spinor-condensate fluctuations to the lm representation

of atomic orbitals. In atomic orbital theory, we know that in the absence of rotational-symmetry breaking all m-states within the same l are degenerate. But in the presence of a crystal field, this degeneracy is lifted. The effect of crystal fields on angular-momentum multiplets is very well-documented (see, e.g., [32]); we can now use this resource to directly find the eigenmodes of the spinor condensates.

Let us demonstrate this principal again using the cyclic state. We have already shown that the $l = 1$ vibration modes correspond to rotations. Let us now consider the $l = 2$ states. Under the effect of a tetrahedral crystal field the electronic states split as:

$$5d \rightarrow \begin{cases} d_{xy}, d_{xz}, d_{yz} \\ d_{z^2}, d_{x^2-y^2} \end{cases} \qquad (82)$$

Now we can map back these atomic states to spinor-condensate oscillation modes. Starting with $d_{xy} \propto Y_{2,2} - Y_{2,-2}$ we find

$$\mathbf{u}_{2,2} - \mathbf{u}_{2,-2} \propto \{1, 1, 1, 1\} \qquad (83)$$

which corresponds to uniform rotation about the z-axis. Similarly $d_{xz} \propto Y_{2,1} - Y_{2,-1}$ and $d_{yz} \propto Y_{2,1} + Y_{2,-1}$ correspond to rotations about the y and x-axis respectively. The two remaining orbitals are $d_{z^2} \propto Y_{2,0}$ and $d_{x^2-y^2} \propto Y_{2,2} + Y_{2,-2}$. Since there are only four independent vectors, $\mathbf{z}$, d_{z^2} and $d_{x^2-y^2}$ translate to the same z_{lm}-vector:

$$\mathbf{u}_{2,0} \propto \mathbf{u}_{2,2} + \mathbf{u}_{2,-2} \propto \{1, 1, -1, -1\} \qquad (84)$$

which is exactly the optical mode shown in Fig. 1.

V. CONCLUSIONS

In this work, we applied the hydrodynamic description developed in Ref. [1] to study the low lying excitations of the spinor condensate in the vicinity of the mean field-ground state. The dynamics of spinor condensates close to the mean-field ground state is where their hidden point-group symmetry becomes most apparent and accessible. Using the spin-node formalism, and the parametrization of the spin-nodes in terms of a stereographic projection, we reduced the problem of finding the $2F$ spin-wave eigenmodes to a simple question of decomposing a representation of the appropriate point symmetry group to its irreducible representations. We also provided a simple recipe that allows the direct extraction of the condensate's spin-wave eigenmodes using the derivatives of the spherical harmonics, coupled with the knowledge of atomic orbital degeneracy lifting under a crystal field. Also, quite generally, we showed that all non-degenerate eigenvectors of the matrix A_{ij} defined in Eq. (49) correspond to the eigenmodes of spinor condensates, independent of the interaction (so long as it preserves $SU(2)$ symmetry).

More than any specific result, the current paper and Ref. [1] derive a new formalism to address high-spin many body systems. It is our impression that, by far, we have not yet explored all possible applications of this formalism. A simple example is the calculation of the spin-wave eigenmodes and energies of a spinor condensate which is locally at its ground state, but with its spin-nodes structure rotated as a function of space. This can be done by combining the linearization of Sec. III with the general hydrodynamic description derived in Sec. II C. Similarly, our method of expanding about a mean-field ground state in terms of the z-variables could be readily applied to computing the leading instabilities in quantum-quench experiments (as in, for instance, Ref. [6] where spin-one quantum quench experiments were performed). The linearized Lagrangian derived in Sec. III C applies near any extremum of the spin interaction energy, V_{int}, even an unstable one. This can then be used to investigate the dynamics for short time-scales after a quantum quench.

Another possible direction focuses on the form of the spin interaction energy $V_{int}(\mathbf{n}_1, \mathbf{n}_2, \dots \mathbf{n}_{2F})$. In terms of the spin-nodes, the spin interaction energy must be a permutation symmetric function of the spin nodes. But the number of permutation symmetric scalars constructed of the spin nodes $\hat{\mathbf{n}}_i$ is limited. All such scalars must be constructed from tensors of the form:

$$\mathcal{M}_{\alpha_1 \alpha_2 \dots \alpha_n} = \sum_{i=1}^{2F} n_{i,\alpha_1} n_{i,\alpha_2} \dots n_{i,\alpha_n}, \qquad (85)$$

where $\alpha_k = x, y, z$ is the space direction. Examples are:

$$\sum_{i,j=1}^{2F} \mathbf{n}_i \cdot \mathbf{n}_j = \sum_{i,j=1}^{2F} n_{i,\alpha} n_{j,\alpha} \\ \sum_{i,j=1}^{2F} n_{i,\alpha} n_{i,\beta} \cdot n_{j,\alpha} n_{j,\beta} \qquad (86)$$

and so forth. This structure of the spin interaction may be used to construct generic phenomenological theories for spinor condensates and other high-spin many-body systems, along the lines of the construction of Landau free energy.

The most interesting applications of the spin-node formalism may arise when considering non-condensed spinor systems. Lattice insulators, both fermionic and bosonic, could also be parametrized using spin-nodes, and should exhibit magnetic mean-field states with hidden point-group symmetry as well. Similarly, we intend to consider spinor *Fermi liquids* using this formalism; such systems may have interesting magnetic instabilities into states with the same hidden symmetries as those arising in spinor condensates.

In the challenging field of many body quantum systems, often a new technical perspective on a problem may simplify it dramatically. In this paper we developed a formalism that seeks to do exactly that to the dynamics of spinor condensates – a topic of much current experimental as well as theoretical interest. Our analysis

provides an economical representation, which allows for a direct, general, and easy calculation of many dynamic collective properties of spinor condensates. In addition, we hope that the developments presented here could be used in other challenging problems involving interacting quantum systems with high spin.

APPENDIX A: THE SPINOR BASIS $\{|T_i\boldsymbol{\Omega}\rangle\}$ AND THE MATRIX A

In this Appendix, we will develop derive identities used for the projection operator $\bar{\mathcal{P}} = 1 - |\chi\rangle\langle\chi|$ and the symmetry matrix A. Consider a particular spinor

$$|\boldsymbol{\Omega}\rangle = |\Omega_1\Omega_2\ldots\Omega_{2F}\rangle \qquad (A1)$$

where none of the spin nodes are degenerate. Then from this we can construct a set of $2F$ states where one of the elements of $|\boldsymbol{\Omega}\rangle$ is time-reversed $\{|T_i\boldsymbol{\Omega}\rangle\}$. Furthermore, we construct the set of $2F$ coherent states which are orthogonal to $|\boldsymbol{\Omega}\rangle$ which are $\left\{\left|(\Omega_i^t)^{2F}\right\rangle\right\}$. We note that these two sets of states satisfy reciprocal relations:

$$\langle T_i\boldsymbol{\Omega}|\,(\Omega_j^t)^{2F}\rangle = \lambda_i\delta_{ij} \qquad (A2)$$

where

$$\lambda_i = (2F)!\prod_{j\neq i}\langle\Omega_j|\Omega_i^t\rangle \qquad (A3)$$

This relation leads to a useful identity for the projection operator

$$\mathcal{P} = \sum_i \frac{\left|(\Omega_i^t)^{2F}\right\rangle\langle T_i\boldsymbol{\Omega}|}{\lambda_i}\mathcal{P}. \qquad (A4)$$

This relation can be immediately proved by expanding any state acting on the right in a basis of states $\left\{\left|(\Omega_i^t)^{2F}\right\rangle\right\}$, and any state acting on the left in a basis of states $\{|T_i\boldsymbol{\Omega}\rangle\}$ (both which, in addition to the state $|\boldsymbol{\Omega}\rangle$, form a complete basis of spinor states when the spin nodes are non-degenerate).

Using these states, we will now proceed to derive an expression for the inverse of the matrix $A_{ij} = \langle T_i\boldsymbol{\Omega}|\,\mathcal{P}\,|T_j\boldsymbol{\Omega}\rangle$ which exists when none of the spin nodes $\mathbf{n}_i$ are degenerate. We define B to be the matrix of the overlap of time-reversed coherent states (which will be shown to be the inverse of A)

$$B_{ij} = \frac{\left\langle(\Omega_i^t)^{2F}\middle|(\Omega_j^t)^{2F}\right\rangle}{\lambda_i^*\lambda_j}. \qquad (A5)$$

Consider the product of these matrices

$$\sum_j B_{ij}A_{jk} = \sum_j \frac{\left\langle(\Omega_i^t)^{2F}\middle|(\Omega_j^t)^{2F}\right\rangle}{\lambda_i^*\lambda_j}\langle T_j\boldsymbol{\Omega}|\,\mathcal{P}\,|T_k\boldsymbol{\Omega}\rangle. \qquad (A6)$$

We can then use the identity in Eq. (A4) to collapse the sum over j. This leads to

$$\sum_j B_{ij}A_{jk} = \frac{\langle(\Omega_i^t)^{2F}|T_k\boldsymbol{\Omega}\rangle}{\lambda_i^*} = \delta_{ik}. \qquad (A7)$$

and the proof is complete.

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